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Alternative analytical forms of the Fuchs–Sondheimer function

C. R. PICHARD, M. BEDDA, V. I. VATAMANYUK*, A. J. TOSSER
Laboratoire d'electronique, Universite de Nancy 1, B.P. 239, 54506 Vandoeuvre-les-Nancy Cedex, France

C. R. TELLIER
Laboratoire de Chronométrie et Piézoélectricité, ENSMM La Bouloie, Route de Gray, 25030 Vesancon Cedix, France

Starting from the Cottey conduction model and its extension, it is shown that the Fuchs–Sondheimer functions can be approximated by the extended Cottey function at any reduced thickness, provided that the specular electronic reflection coefficient, *p*, takes values larger than 0.31. Whatever the values of *p* and the film thickness be an analytical formulation (in the form of the Cottey function) is proposed for an accurate approximation of the Fuchs–Sondheimer function. Moreover, it is suggested that the scattering processes defined in the Fuchs–Sondheimer model have no interaction.

1. Introduction

Numerical data [1, 2] related to the Fuchs–Sondheimer conduction model [3] (the F–S model), and the extended form [1] of the Cottey conduction model [4] (the e–C model), have shown that the deviation in the reduced resistivity is slight when the electron specular reflection coefficient at film surface, p, takes values larger than 0.3 and when the film reduced thickness is large.

It is attempted in this paper to give a theoretical basis for this feature and to propose new analytical forms whose validity range is more extended than that of previous approximate equations.

2. Preliminary results

2.1. Deviations in the e–C model from the F–S model

Extended calculations [5] of the reduced electrical resistivity, ϱ_f/ϱ_0 (where the index f and 0 is related to the film and the bulk material, respectively (Table I) show that the deviation from the F–S model is less than 10% for $p \geqslant 0.3$ and $k > 0.1$ where k is the reduced thickness, defined by

$$k = d\lambda_0^{-1}$$

where d is the film thickness, λ_0 the electron mean free path in the bulk material and μ the size parameter of the effective Cottey model defined by

$$\mu = k\left(\ln \frac{1}{p}\right)^{-1}$$

where p is the electron specular reflection coefficient at film surface.

The deviation depends mainly on the value of p and increases as the value of p decreases (Fig. 1).

2.2. Theoretical expressions for the reduced resistivity in the F–S model

The reduced conductivity, σ_f/σ_0, is usually given in the following form ([3], Equations 25)

$$\sigma_f/\sigma_0 = 1 - A(k, p) \quad (1)$$

*Permanent address: Physics Institute, Ukrainian Academy of Sciences, 144 pr. Nauki-Kiev-28, USSR.

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TABLE I Comparison of variations in the reduced resistivity of thin monocrystal films ϱ_f/ϱ_0 with film reduced thickness, k, and specular reflection coefficient p, in the framework of the extended Cottey model, $\varrho_f/\varrho_o = [C(\mu)]^{-1}$, and of the Fuchs–Sondheimer conduction model $\varrho_f/\varrho_0 = F(k, p)$

$p = 0.25$				$p = 0.5$				$p = 0.75$			
k	$1/C(\mu)$	$F(K, P)$	$-[(F - 1/C)/F](\%)$	k	$1/C(\mu)$	$F(K, P)$	$-[(F - 1/C)/F](\%)$	k	$1/C(\mu)$	$F(K, P)$	$-[(F - 1/C)/F](\%)$
0.01	20.7873	18.1950	14.2	0.01	12.2685	11.8200	3.7	0.01	6.5588	6.5149	0.6
0.02	12.2685	10.7600	14.0	0.02	7.4535	7.1839	3.7	0.02	4.1881	4.1603	0.6
0.03	9.1293	8.0187	13.8	0.03	5.6635	5.4601	3.7	0.03	3.2979	3.2760	0.6
0.04	7.4535	6.5548	13.7	0.04	4.7028	4.5350	3.7	0.04	2.8178	2.7992	0.6
0.05	6.3969	5.6317	13.5	0.05	4.0948	3.9496	3.6	0.05	2.5132	2.4968	0.6
0.06	5.6635	4.9909	13.4	0.06	3.6717	3.5422	3.6	0.06	2.3011	2.2860	0.6
0.07	5.1215	4.5174	13.3	0.07	3.3585	3.2406	3.6	0.07	2.1440	2.1300	0.6
0.08	4.7028	4.1516	13.2	0.08	3.1161	3.0073	3.6	0.08	2.0225	2.0093	0.6
0.09	4.3685	3.8597	13.1	0.09	2.9224	2.8209	3.6	0.09	1.9254	1.9130	0.6
0.10	4.0948	3.6206	13.0	0.10	2.7637	2 6681	3.5	0.10	1.8460	1.8341	0.6
0.20	2.7637	2.4593	12.3	0.20	1.9911	1.9250	3.4	0.20	1.4623	1.4532	0.6
0.30	2.2617	2.0230	11.8	0.30	1.7007	1.6463	3.3	0.30	1.3208	1.3128	0.6
0.40	1.9911	1.7889	11.3	0.40	1.5452	1.4974	3.1	0.40	1.2462	1.2389	0.5
0.50	1.8197	1.6415	10.8	0.50	1.4473	1.4040	3.0	0.50	1.2000	1.1932	0.5
0.60	1.7007	1.5398	10.4	0.60	1.3798	1.3398	2.9	0.60	1.1685	1.1620	0.5
0.70	1.6129	1.4652	10.0	0.70	1.3303	1.2929	2.8	0.70	1.1456	1.1394	0.5
0.80	1.5452	1.4081	9.7	0.80	1.2924	1.2571	2.8	0.80	1.1282	1.1223	0.5
0.90	1.4913	1.3630	9.4	0.90	1.2623	1.2289	2.7	0.90	1.1145	1.1088	0.5
1.00	1.4473	1.3265	9.1	1.00	1.2380	1.2061	2.6	1.00	1.1035	1.0980	0.5
2.00	1.2380	1.1585	6.8	2.00	1.1237	1.1013	2.0	2.00	1.0527	1.0487	0.3
3.00	1.1627	1.1026	5.4	3.00	1.0837	1.0663	1.6	3.00	1.0354	1.0321	0.3
4.00	1.1237	1.0754	4.5	4.00	1.0633	1.0491	1.3	4.00	1.0266	1.0239	0.2
5.00	1.0999	1.0595	3.8	5.00	1.0509	1.0389	1.1	5.00	1.0213	1.0191	0.2
6.00	1.0837	1.0491	3.3	6.00	1.0425	1.0322	1.0	6.00	1.0178	1.0158	0.2
7.00	1.0721	1.0418	2.9	7.00	1.0365	1.0275	0.8	7.00	1.0153	1.0135	0.1
8.00	1.0633	1.0364	2.5	8.00	1.0320	1.0239	0.7	8.00	1.0134	1.0118	0.1
9.00	1.0564	1.0322	2.3	9.00	1.0285	1.0212	0.7	9.00	1.0119	1.0105	0.1
10.00	1.0509	1.0289	2.1	10.00	1.0257	1.0191	0.6	10.00	1.0107	1.0094	0.1

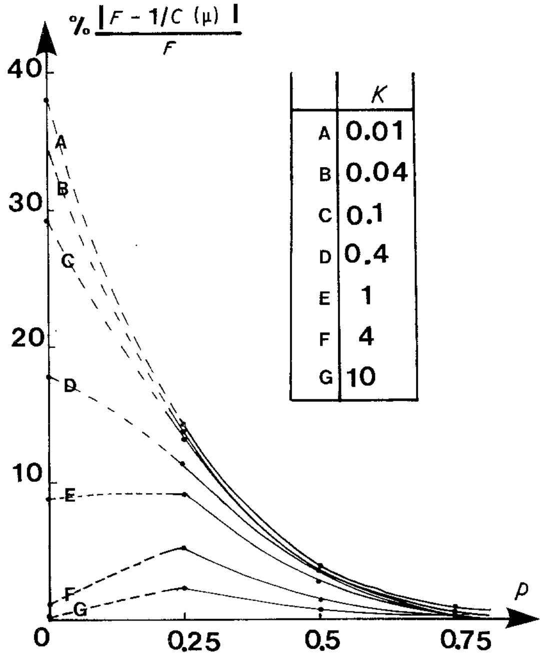

Figure 1 The relative denotion $|F - 1/C(\mu)|/F$ with the specular parameter, p, for different values of the reduced thickness, k.

with

$$A(k, p) = \frac{3}{2k}\int_1^{\infty}\left(\frac{1}{t^3} - \frac{1}{t^5}\right)\frac{1 - \exp(-kt)}{1 - p\exp(-kt)}\mathrm{d}t$$

$$k = d\lambda_0^{-1} \qquad (2)$$

where t is an integration variable.

Whatever the value of p, the following asymptotic F–S expressions ([3], Equations 26 and 27) can be used for the reduced resistivity

$$\varrho_f/\varrho_0 \approx \left[1 + \frac{3}{8k}(1 - p)\right], \qquad k \gg 1 \; (3a)$$

which can also be written as

$$\varrho_f/\varrho_0 \approx \left[1 - \frac{3}{8k}(1 - p)\right]^{-1}, \qquad k \gg 1 \qquad (3b)$$

and

$$\varrho_f/\varrho_0 \approx \left[\frac{3}{4}\frac{1 + p}{1 - p}k\ln\frac{1}{k}\right]^{-1}, \qquad k \ll 1 \quad (4)$$

2.3. Theoretical expressions for the reduced conductivity in the e–C model

The reduced conductivity is expressed in terms of the Cottey function $C(\mu)$ [1, 4]

$$\sigma_f/\sigma_0 = C(\mu) \qquad (5)$$

with

$$C(\mu) = \tfrac{3}{4}\mu[\mu - \tfrac{1}{2} + (1 - \mu^2)\ln(1 + \mu^{-1})] \qquad (6)$$

In the framework of the e–C model, the parameter μ is expressed as ([1] Equation 1.85, [2])

$$\mu = k\left(\ln\frac{1}{p}\right)^{-1}, \qquad p \neq 0 \qquad (7)$$

In the limiting cases of large and low thickness the approximate expressions are [1, 4]

$$\sigma_f/\sigma_0 \approx 1 - \frac{3}{8k}\ln\frac{1}{p}, \qquad p \neq 0,\ k \gg 1 \, (8a)$$

or

$$\varrho_f/\varrho_0 \approx 1 - \frac{3}{8k}\ln\frac{1}{p}, \qquad p \neq 0,\ k \gg 1 \quad (8)$$

and

$$\varrho_f/\varrho_0 \approx [\tfrac{3}{4}\mu\ln 1/\mu]^{-1}, \qquad p \neq 0,\ \mu \ll 1 \qquad (9a)$$

i.e.

$$\varrho_f/\varrho_0 \approx \left[\tfrac{3}{4}k\ln\left(\ln\frac{1}{p}\right)^{-1}\right]^{-1}, \qquad p \neq 0,\ k \ll 1 \qquad (9b)$$

since $1/p$ cannot take infinite values in the framework of the e–C model (see Equation 7).

2.4. Ranges of validity of the approximate F–S equations

Numerical data (Table II) shows that the exact F–S equations cannot always be obtained to a good accuracy by using one of the asymptotic equations defined in the above paragraph; for instance, at low reduced thickness when p takes values lower than 0.5, no approximate equation is valid.

Consequently, no analytical expressions is available for approximating the F–S equations in the whole experimental field.

3. Comparative studies

3.1. The asymptotic formulae

Equations 3a and 8b coincide if

$$\ln\frac{1}{p} \approx (1 - p) \qquad (10)$$

TABLE II Numerical values for the exact (Equations 1) and approximate F–S equations (Equations 3a and 4) for a set of values of the electronic specular reflection coefficient, p, and of the reduced thickness, k

$p = 0$				$p = 0.25$			
K	F(Equation 1)	F(Equation 3a)	F(Equation 4)	K	F(Equation 1)	F(Equation 3a)	F(Equation 4)
0.01	26.4830	38.5000	+28.9529	0.01	18.1950	29.1250	+17.3717
0.02	15.3320	19.7500	+17.0414	0.02	10.7600	15.0625	+10.2248
0.03	11.2540	13.5000	+12.6746	0.03	8.0187	10.3750	+7.6047
0.04	9.0885	10.3750	+10.3555	0.04	6.5548	8.0312	+6.2133
0.05	7.7276	8.5000	+8.9015	0.05	5.6317	6.6250	+5.3409
0.06	6.7859	7.2500	+7.8986	0.06	4.9909	5.6875	+4.7392
0.07	6.0917	6.3571	+7.1627	0.07	4.5174	5.0178	+4.2976
0.08	5.5566	5.6875	+6.5987	0.08	4.1516	4.5156	+3.9592
0.09	5.1302	5.1666	+6.1524	0.09	3.8597	4.1250	+3.6914
0.10	4.7816	4.7500	+5.7905	0.10	3.6206	3.8125	+3.4743
0.20	3.0958	2.8750	+4.1422	0.20	2.4593	2.4062	+2.4853
0.30	2.4658	2.2500	+3.6914	0.30	2.0230	1.9375	+2.2148
0.40	2.1284	1.9375	+3.6378	0.40	1.7889	1.7031	+2.1827
0.50	1.9161	1.7500	+3.8471	0.50	1.6415	1.5625	+2.3083
0.60	1.7695	1.6250	+4.3502	0.60	1.5398	1.4687	+2.6101
0.70	1.6621	1.5357	+5.3403	0.70	1.4652	1.4017	+3.2041
0.80	1.5798	1.4687	+7.4690	0.80	1.4081	1.3515	+4.4814
0.90	1.5148	1.4166	+14.0610	0.90	1.3630	1.3125	+8.4366
1.00	1.4622	1.3750	-3.6092×10^9	1.00	1.3265	1.2812	-2.1655×10^9
2.00	1.2208	1.1875	−0.9617	2.00	1.1585	1.1406	−0.5770
3.00	1.1414	1.1250	−0.4045	3.00	1.1026	1.0937	−0.2427
4.00	1.1031	1.0937	−0.2404	4.00	1.0754	1.0703	−0.1442
5.00	1.0810	1.0750	−0.1656	5.00	1.0595	1.0562	−0.0994
6.00	1.0666	1.0625	−0.1240	6.00	1.0491	1.0468	−0.0744
7.00	1.0566	1.0535	−0.0978	7.00	1.0418	1.0401	−0.0587
8.00	1.0491	1.0468	−0.0801	8.00	1.0364	1.0351	−0.0480
9.00	1.0434	1.0416	−0.0674	9.00	1.0322	1.0312	−0.0404
10.00	1.0389	1.0375	−0.0579	10.00	1.0289	1.0281	−0.0347

$p = 0.5$				$p = 0.75$			
K	F(Equation 1)	F(Equation 3a)	F(Equation 4)	K	F(Equation 1)	F(Equation 3a)	F(Equation 4)
0.01	11.8200	19.7500	+9.6509	0.01	6.5149	10.3750	+4.1361
0.02	7.1839	10.3750	+5.6804	0.02	4.1603	5.6875	+2.4344
0.03	5.4601	7.2500	+4.2248	0.03	3.2760	4.1250	+1.8106
0.04	4.5350	5.6875	+3.4518	0.04	2.7992	3.3437	+1.4793
0.05	3.9496	4.7500	+2.9671	0.05	2.4968	2.8750	+1.2716
0.06	3.5422	4.1250	+2.6328	0.06	2.2860	2.5625	+1.1283
0.07	3.2406	3.6785	+2.3875	0.07	2.1300	2.3392	+1.0232
0.08	3.0073	3.3437	+2.1995	0.08	2.0093	2.1718	+0.9426
0.09	2.8209	3.0833	+2.0508	0.09	1.9130	2.0416	+0.8789
0.10	2.6681	2.8750	+1.9301	0.10	1.8341	1.9375	+0.8272
0.20	1.9250	1.9375	+1.3807	0.20	1.4532	1.4687	+0.5917
0.30	1.6463	1.6250	+1.2304	0.30	1.3128	1.3125	+0.5273
0.40	1.4974	1.4687	+1.2126	0.40	1.2389	1.2343	+0.5196
0.50	1.4040	1.3750	+1.2823	0.50	1.1932	1.1875	+0.5495
0.60	1.3398	1.3125	+1.4500	0.60	1.1620	1.1562	+0.6214
0.70	1.2929	1.2678	+1.7801	0.70	1.1394	1.1339	+0.7629
0.80	1.2571	1.2343	+2.4896	0.80	1.1223	1.1171	+1.0670
0.90	1.2289	1.2083	+4.6870	0.90	1.1088	1.1041	+2.0087
1.00	1.2061	1.1875	-1.203×10^9	1.00	1.0980	1.0937	-5.156×10^8
2.00	1.1013	1.0937	−0.3205	2.00	1.0487	1.0468	−0.1373
3.00	1.0663	1.0625	−0.1348	3.00	1.0321	1.0312	−0.0577
4.00	1.0491	1.0468	−0.0801	4.00	1.0239	1.0234	−0.0343
5.00	1.0389	1.0375	−0.0552	5.00	1.0191	1.0187	−0.0236
6.00	1.0322	1.0312	−0.0413	6.00	1.0158	1.0156	−0.0177
7.00	1.0275	1.0267	−0.0326	7.00	1.0135	1.0133	−0.0139
8.00	1.0239	1.0234	−0.0267	8.00	1.0118	1.0117	−0.0114
9.00	1.0212	1.0208	−0.0224	9.00	1.0105	1.0104	−0.0096
10.00	1.0191	1.0187	−0.0193	10.00	1.0094	1.0093	−0.0082

Similarly, from Equations 4 and 9b

$$\ln\frac{1}{p} \approx 2\frac{1-p}{1+p} \quad (11)$$

It is clear that Equation 10 is satisfied when p takes values in the vicinity of unity; moreover Equation 11 reduces to Equation 10 in this case. Consequently Equation 11 is the only condition for the coincidence of the asymptotic Equations 3, 4, 8 and 9.

From extended calculations of $[\ln(1/p)]^{-1}$ and $(1+p)/2(1-p)$ (Table III), it is seen that Equation 11 is valid down to $p = 0.31$ with a relative deviation less than 10%.

TABLE III Numerical values of the functions $A = [\ln(1/p)]^{-1}$, $B = (1+p)/2(1-p)$ and $(B-A)/B$ for $0.1 \leqslant p \leqslant 0.98$

p	$[\log(1/p)]^{-1}$ (A)	$(1+p)/[2(1-p)]$ (B)	$-(A-B)/B(\%)$
0.10	0.4342	0.6111	28.93
0.12	0.4716	0.6363	25.88
0.14	0.5086	0.6627	23.26
0.16	0.5456	0.6904	20.97
0.18	0.5831	0.7195	18.95
0.20	0.6213	0.7500	17.15
0.22	0.6604	0.7820	15.54
0.24	0.7007	0.8157	14.10
0.26	0.7423	0.8513	12.80
0.28	0.7855	0.8888	11.62
0.30	0.8305	0.9285	10.55
0.32	0.8776	0.9705	9.57
0.34	0.9269	1.0151	8.68
0.36	0.9788	1.0625	7.87
0.38	1.0335	1.1129	7.13
0.40	1.0913	1.1666	6.45
0.42	1.1527	1.2241	5.83
0.44	1.2180	1.2857	5.26
0.46	1.2877	1.3518	4.73
0.48	1.3624	1.4230	4.25
0.50	1.4426	1.5000	3.82
0.52	1.5292	1.5833	3.41
0.54	1.6228	1.6739	3.04
0.56	1.7246	1.7727	2.71
0.58	1.8357	1.8809	2.40
0.60	1.9576	2.0000	2.11
0.62	2.0918	2.1315	1.86
0.64	2.2407	2.2777	1.62
0.66	2.4066	2.4411	1.41
0.68	2.5929	2.6250	1.22
0.70	2.8036	2.8333	1.04
0.72	3.0441	3.0714	0.88
0.74	3.3210	3.3461	0.74
0.76	3.6438	3.6666	0.62
0.78	4.0247	4.0454	0.51
0.80	4.4814	4.5000	0.41
0.82	5.0390	5.0555	0.32
0.84	5.7354	5.7500	0.25
0.86	6.6302	6.6428	0.18
0.88	7.8226	7.8333	0.13
0.90	9.4912	9.5000	0.09
0.92	11.9930	12.0000	0.05
0.94	16.1615	16.1666	0.03
0.96	24.4965	24.5000	0.01
0.98	49.4983	49.5000	0.00

3.2. Unattempted consequences

The above results show that in the whole range of thicknesses, practically, the F–S model and the e–C model coincide, provided that p takes a value larger than 0.31.

This feature is somewhat surprising because the theoretical bases for these models differ. Let us remember that the F–S model uses the Boltzmann equation for the charge transport, whereas the e–C model [1] (like the Cottey model [4]) assumes that a resultant mean free path, λ, may be calculated by adding the effects of background scattering and external surface scattering. More precisely

$$\lambda_r^{-1} = \lambda_0^{-1} + \lambda_c^{-1} \quad (12)$$

where λ_c is the Cottey mean free path describing the electron scattering at film surface [4, 1].

Some consequences can be derived, which contradict several opinions which are sometimes presented.

1. Even at low film thickness, the e–C model can be regarded as equivalent to the F–S model, provided that $p > 0.31$.

2. The only limit for the validity of the e–C model is $p > 0.31$ and no limitation in thickness can be retained, excluding the case of film thickness having a magnitude similar to that of the wavelength of Fermi electrons which cannot be treated by the Fuchs–Sondheimer procedure [3, 6].

3. A further consequence of (1) is the following: in the F–S model no marked interaction of the two types of electron scattering (background and external surface) can be clearly pointed out in the whole range of thickness, provided that $p > 0.3$.

3.3. Discussion

Starting from the above conclusions the reason for the inadequacy of the e–C model at low values of p is the fact that the e–C model (like the Cottey model) defines a continuously variable probablity of the distribution of electron paths, whereas the F–S model is based on boundary conditions (at well-defined geometrical positions) (Fig. 2). It is clear that the F–S curve cannot be accurately represented by the exponential e–C curve when the alteration in the distribution function of electrons is too marked at any scattering.

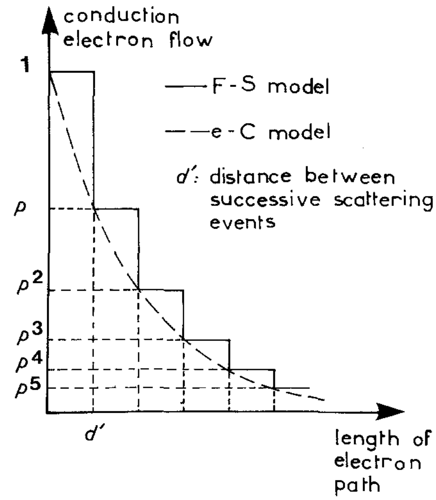

Figure 2 Distribution of electron paths in the F–S and e–C models.

No interaction of the two types of scattering is clearly revealed in the framework of the F–S model (from Equations 1 and 2), but one must not forget that the boundary conditions for the perturbation of the electron distribution do depend on both background and external surface scatterings ([3] Equation 24); they are introduced in the general expression of the current density which depends on the z-coordinate (the z-axis being perpendicular to the film substrate). For a comparison with experiments the average value of the current density over the film thickness is calculated [3], that yields Equations 1 and 2.

It can be easily assumed that the averaging procedure masks the interaction of the scatterings at large film thickness because most of the conduction electrons are not scattered at film surfaces in this case; but this interpretation is not valid in the case of thin films and an open problem remains.

The statistical effects of two scattering phenomena on an electron along its path of length l (Fig. 3) can be treated in the same way

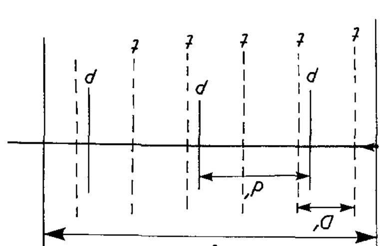

Figure 3 Geometry of statistical model in the case of grain-boundary and external surfaces scatterings.

as in the statistical models [1, 7] used for the description of grain boundary and external surface scatterings [1, 8, 9]; it shows that the resultant statistical transmission coefficient of electron flow, t_r, can be calculated from the statistical coefficient of the first scattering procedure, t, and from p (as in the Cottey model [4, 9]); hence, in the case of Fig. 3

$$t_r = t^6 p^3$$
$$\approx \exp\left(-\frac{l}{D'}\ln\frac{1}{t}\right)\exp\left(-\frac{1}{d'}\ln\frac{1}{p}\right) \quad (13)$$

where D' and d' are the distance between successive scattering of each type in the direction of electron motion.

It is clear that Equation 13 yields

$$t_r = \exp(-l/\lambda_r) \quad (14)$$

where λ_r^{-1} is given by an equation similar to Equation 12, with

$$\lambda_1 = D'/\ln\frac{1}{t}; \qquad \lambda_c = d'/\ln\frac{1}{p} \quad (15)$$

In such a statistical treatment the interaction of scattering phenomena does not exist because the background scattering is defined in the same form as the grain boundary scattering [1, 7].

The averaging procedure of the F–S model is not far from the above procedure since it consists in substituting an average value for the z-dependent value of the distribution perturbation at any level z (the film extends from $z = 0$ to $z = d$).

This point is now emphasized.

4. Analysis of the calculation procedure of Sondheimer [3]

4.1. Theory

For calculating the electrical conductivity of thin metal films, σ_f, Sondheimer solved the Boltzmann equation, assuming a variation in the distribution function of electrons, f, with z (thickness direction) and neglecting the product of the electric field with the gradient of the deviation f_1 of the distribution function f from its equilibrium value f_0.

In the case of an electric field, E, in the direction of the x-axis, the Boltzmann equation reduces to ([3] Equation 9)

$$\frac{\delta f_1}{\delta z} + \frac{f_1}{\tau v_z} = \frac{eE}{mv_z}\frac{\delta f_0}{\delta v_x} \quad (16)$$

where v is the electron velocity, τ its relaxation time and e its absolute charge.

Integrating Equation 16 and taking into account the distribution functions of the electrons leaving each external surface ([3] Equations 22 and 23), the current density $J(z)$ may be written as

$$J(z) = \frac{4\pi e^2 m^2 \tau \bar{V}^3}{h^3} E \int_0^{\pi/2} \sin^3\theta \times \left[1 - \frac{1-p}{1 - p\exp(-d/\tau\bar{V}\cos\theta)} \times \exp\left(-\frac{z}{\tau\bar{V}\cos\theta}\right)\right] d\theta \quad (17)$$

where h is Planck's constant, m the electron effective mass and θ the angle of v with the x-axis.

For the sake of comparison, with experience Sondheimer used the average current density over all values of z from 0 to d for calculating the film conductivity σ ([3] Equation 15)

$$\sigma = \frac{1}{Ed}\int_0^d J(z)\mathrm{d}z \quad (18)$$

The only term which depends on z in Equation 17 is in the integrand; averaging this term gives

$$\frac{1}{d}\int_0^d \exp\left(-\frac{z}{\tau\bar{V}\cos\theta}\right)\mathrm{d}z = \frac{\tau\bar{V}\cos\theta}{d}\left[1 - \exp\left(-\frac{d}{\tau\bar{V}\cos\theta}\right)\right] \quad (19)$$

Equation 17 then becomes

$$\overline{J(z)} = \frac{4\pi e^2 m^2 \tau \bar{V}^3}{h^3} E \int_0^{\pi/2} \sin^3\theta \times \left\{1 - \frac{1-p}{1 - p\exp(d/\tau\bar{V}\cos\theta)} \times \frac{\tau\bar{V}\cos\theta}{d} \times \left[1 - \exp\left(-\frac{d}{\tau\bar{V}\cos\theta}\right)\right]\right\} d\theta \quad (20)$$

Introducing for convenience the conductivity of the bulk metal, σ_0, and the reduced thickness, $k = d\lambda_0^{-1}$, Equation 18 takes the following form

$$\sigma_f/\sigma_0 = \tfrac{3}{2}\int_0^{\pi/2} \sin^3\theta \times \left[1 - \frac{1-p}{k}\cos\theta\,\frac{1 - \exp(-k/\cos\theta)}{1 - p\exp(-k/\cos\theta)}\right] d\theta \quad (21)$$

Integrating by parts and defining the integration variable t by the relation

$$t = \frac{1}{\cos\theta}$$

Equation 21 can be rewritten in the usual form ([3] Equations 16 and 25) of Equation 2.

An effective mean free path, $\lambda_{\mathrm{eff}}(\theta)$, can thus be defined from the relation

$$\lambda_{\mathrm{eff}}(\theta) = \lambda_0\left[1 - \frac{1-p}{k}\cos\theta\frac{1-\exp(-k/\cos\theta)}{1-p\exp(-k/\cos\theta)}\right] \tag{22}$$

so that Equation 21 becomes

$$\sigma_{\mathrm{f}}/\sigma_0 = \frac{3}{2}\int_0^{\pi/2}\frac{\lambda_{\mathrm{eff}}(\theta)}{\lambda_0}\sin^3\theta\,\mathrm{d}\theta \tag{23}$$

as attempted [1].

In order to examine the variations in $\lambda_{\mathrm{eff}}(\theta)$ with k it is convenient to assume that $(\cos\theta)^{-1}$ takes a finite value, i.e. $\theta \neq \pi/2$.

4.2. Comparing F–S and e–C models for $p > 0.3$

At large film thickness, Equation 22 reduces to

$$\lambda_{\mathrm{eff}}(\theta) \approx \lambda_0\left(1 - \frac{\alpha\cos\theta}{k}\right), \qquad k \gg 1 \tag{24a}$$

with

$$p = 1 - \alpha, \qquad \alpha \ll 1$$

An alternative form is

$$\lambda_{\mathrm{eff}}(\theta) \approx \lambda_0\left(1 + \alpha\frac{\cos\theta}{k}\right)^{-1}, \qquad k \gg 1, \quad p \approx 1 \tag{24b}$$

Equation 24b is also obtained in the framework of the e–C model [1], assuming that the resultant mean free path, λ_{r}, due to the two types of scatterings (background and external surface) is expressed from the bulk mean free path, λ_0, and from the Cottey mean free path, λ_{c}, by Equation 12

$$\lambda_{\mathrm{r}}^{-1} = \lambda_0^{-1} + \lambda_{\mathrm{c}}^{-1} \tag{25a}$$

with [1],

$$\lambda_{\mathrm{c}} = d|\cos\theta|^{-1}\left(\ln\frac{1}{p}\right)^{-1} \tag{25b}$$

Hence

$$\lambda_{\mathrm{r}} = \lambda_0\left(1 + \lambda_0\cos\theta\ln\frac{1}{p}d^{-1}\right)^{-1} \tag{26}$$

When p takes values near unity, Equation 25b goes to the approximate form

$$\lambda_{\mathrm{c}} \approx d|\cos\theta|^{-1}(1-p)^{-1} \tag{27}$$

Hence Equation 12 becomes

$$\lambda_{\mathrm{r}} \approx \lambda_0\left[1 + (1-p)\frac{|\cos\theta|}{k}\right]^{-1}, \qquad p \approx 1 \tag{28}$$

in good agreement with Equation 24b.

A theoretical basis is thus given for the similar behaviour of the electrical conductivity of thin films observed [1] in the frameworks of the extended Cottey model and of the Fuchs–Sondheimer model.

At low thickness, Equation 22 can be expanded as follows

$$\lambda_{\mathrm{eff}}(\theta) \approx \lambda_0\left[1 - \frac{1-p}{k}\cos\theta \times\left(\frac{k}{\cos\theta} - \frac{k^2}{2\cos 2\theta} + \ldots\right) \times\left(1 - p + p\frac{k}{\cos\theta} - \ldots\right)^{-1}\right], \qquad \frac{k}{\cos\theta} \ll 1 \tag{29}$$

it yields

$$\lambda_{\mathrm{eff}} \approx \lambda_0\frac{k}{\cos\theta}\frac{1+p}{2(1-p)}, \qquad \frac{k}{\cos\theta} \gg 1 \tag{30}$$

At low thickness Equation 12 becomes

$$\lambda_{\mathrm{r}}^{-1} \approx \lambda_{\mathrm{c}}^{-1}, \qquad \frac{k}{\cos\theta} \ll 1 \tag{31}$$

Hence, from Equation 25b

$$\lambda_{\mathrm{r}} \approx \lambda_0\frac{k}{\cos\theta}\left(\ln\frac{1}{p}\right)^{-1}, \qquad \frac{k}{\cos\theta} \ll 1$$

Since numerical values of $[\ln(1/p)]^{-1}$ and $[(1+p)/2(1-p)]$ are very close for $p > 0.31$ it can be deduced from Equations 30 and 32 that

$$\lambda_{\mathrm{eff}} \approx \lambda_{\mathrm{r}} \qquad \frac{k}{\cos\theta} \ll 1$$

In the case where $\theta = \pi/2$, no scattering at film surface occurs nor scattering interaction.

Since the asymptotic values of λ_{eff} and λ_r coincide, the results of Section 2 suggest that it could exist no difference between the effective mean free path $\lambda_{eff}(\theta)$ (which is an alternative presentation of the F–S model) and the resultant mean free path of the e–C model. For this purpose the numerical values of the size function in Equations 12 and 22 are compared.

The tabulated values (Table IV) of the functions

$$f_{FS}(u) = 1 - \frac{1-p}{u}\frac{1-e^{-u}}{1-pe^{-u}} \quad (33)$$

and

$$f_{eC}(u) = \left(1 + \frac{\ln 1/p}{u}\right)^{-1} \quad (34)$$

do establish the close similarity of the mean free paths λ_{eff} and λ_r for $p > 0.3$. Consequently the e–C model is completely identical to the F–S model at any thickness ($0.001 < k < \infty$), provided that $p > 0.3$.

4.3. The case where $p < 0.3$

For low values of p an approximate law for the variations in the electron flow $\phi(l)$ with the electron-path length, l, could be a Gaussian law (Fig. 4) with a standard deviation, $\xi(p)$, which depends on p, i.e.

$$\phi(l) = \exp - (l^2/2\xi^2(p) \quad (35a)$$

with

$$\xi(p) = C(p)\lambda_c' \quad (35b)$$

where λ_c' is the Gaussian mean free path, independent of p.

For the sake of simplicity for further calculations the usual mean free path, λ_c', can be also be defined from the usual exponential law

$$\phi(l) = \exp - (l/\lambda_c') \quad (36)$$

under the condition that the values of $\phi(l)$ in the Gaussian and exponential law coincide when the length l is equal to λ_c'. Consequently

$$C(p)\lambda_c'(2)^{\frac{1}{2}} = \lambda_c' \quad (37)$$

The value of λ_c' is calculated from the value of the electron flow at the first scattering, i.e. for $l = L_x$ (Fig. 4). To a first approximation, this

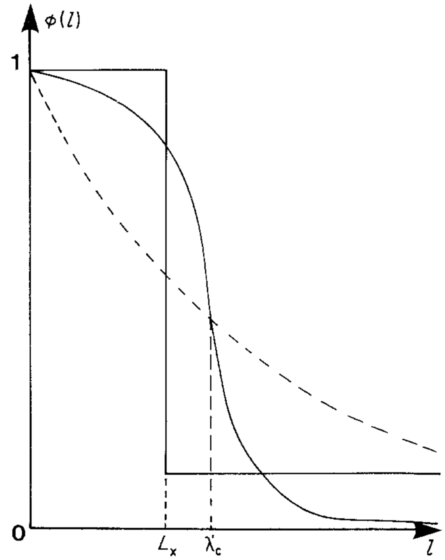

Figure 4 Use of a Gaussian law for representing the variations with path length, l, in the electron flow, $\phi(l)$, at low values of the electronic reflection coefficient, p.

value is taken as a linear function of p. Hence

$$\exp - \frac{L_x^2}{2C(p)^2\lambda_c'^2} = C_2p + C_4 \quad (38)$$

where C_3 and C_4 are constants.

Introducing Equation 38 into Equation 37 then gives

$$\lambda_c' = L_x[\ln(C_3p + C_4)^{-1}]^{-1/2} \quad (39)$$

For low values of p, an approximate form for λ_c' is

$$\lambda_c' \approx L_x\left(\ln\frac{1}{C_4}\right)^{-1/2} \times \left[1 - \left(2\ln\frac{1}{C_4}\right)^{-1}\frac{C_3}{C_4}p\right], \quad p \ll 1 \quad (40a)$$

where (Fig. 4)

$$L_x = d|\cos\theta|^{-1} \quad (40b)$$

The form of Equation 40a is similar to that of Equation 30 because

$$2(1-p)(1+p)^{-1} \approx 2(1-2p), \quad p \ll 1 \quad (41)$$

Consequently, for the sake of simplicity we then examine the following new expression for

TABLE IV Variations in the numerical values of the functions $f_{FS}(u)$ (Equations 33) and $f_{eC}(u)$ (Equation 34) with the parameter u and the electronic reflection coefficient p

$p = 0.3$				$p = 0.25$			
u	f_{eC} (Equation 34)	f_{eC} (Equation 33)	$-(f_{eC} - f_{FS})/(f_{FS}(\%))$	u	f_{eC} (Equation 34)	f_{eC} (Equation 33)	$-(f_{eC} - f_{FS})/(f_{FS}(\%))$
0.001	0.0008	0.0009	10.55	0.001	0.0007	0.0008	13.44
0.002	0.0016	0.0018	10.55	0.002	0.0014	0.0016	13.43
0.003	0.0024	0.0027	10.55	0.003	0.0021	0.0024	13.43
0.004	0.0033	0.0037	10.54	0.004	0.0028	0.0033	13.43
0.005	0.0041	0.0046	10.54	0.005	0.0035	0.0041	13.43
0.006	0.0049	0.0055	10.54	0.006	0.0043	0.0049	13.43
0.007	0.0057	0.0064	10.54	0.007	0.0050	0.0058	13.43
0.008	0.0066	0.0073	10.54	0.008	0.0057	0.0066	13.43
0.009	0.0074	0.0082	10.54	0.009	0.0064	0.0074	13.42
0.010	0.0082	0.0092	10.54	0.010	0.0071	0.0082	13.42
0.020	0.0163	0.0182	10.53	0.020	0.0142	0.0164	13.41
0.030	0.0243	0.0271	10.53	0.030	0.0211	0.0244	13.40
0.040	0.0321	0.0359	10.52	0.040	0.0280	0.0323	13.39
0.050	0.0398	0.0445	10.51	0.050	0.0348	0.0401	13.38
0.060	0.0474	0.0530	10.50	0.060	0.0414	0.0478	13.37
0.070	0.0549	0.0613	10.50	0.070	0.0480	0.0554	13.36
0.080	0.0623	0.0696	10.49	0.080	0.0545	0.0629	13.35
0.090	0.0695	0.0777	10.48	0.090	0.0609	0.0703	13.34
0.100	0.0766	0.0856	10.47	0.100	0.0672	0.0776	13.33
0.200	0.1424	0.1589	10.40	0.200	0.1260	0.1452	13.22
0.300	0.1994	0.2224	10.32	0.300	0.1779	0.2047	13.11
0.400	0.2493	0.2778	10.24	0.400	0.2239	0.2574	13.00
0.500	0.2934	0.3266	10.15	0.500	0.2650	0.3043	12.89
0.600	0.3325	0.3698	10.07	0.600	0.3020	0.3463	12.77
0.700	0.3676	0.4084	9.99	0.700	0.3355	0.3841	12.66
0.800	0.3992	0.4430	9.90	0.800	0.3659	0.4184	12.54
0.900	0.4277	0.4743	9.81	0.900	0.3936	0.4495	12.42
1.000	0.4537	0.5026	9.72	1.000	0.4190	0.4778	12.31
2.000	0.6242	0.6845	8.81	2.000	0.5906	0.6643	11.10
3.000	0.7136	0.7749	7.91	3.000	0.6839	0.7594	9.94
4.000	0.7686	0.8272	7.08	4.000	0.7426	0.8150	8.89
5.000	0.8059	0.8606	6.35	5.000	0.7829	0.8507	7.97
6.000	0.8328	0.8835	5.73	6.000	0.8123	0.8752	7.18
7.000	0.8532	0.9000	5.20	7.000	0.8346	0.8929	6.52
8.000	0.8691	0.9125	4.74	8.000	0.8523	0.9062	5.95
9.000	0.8820	0.9222	4.36	9.000	0.8665	0.9166	5.47
10.000	0.8925	0.9300	4.02	10.000	0.8782	0.9250	5.05
20.000	0.9432	0.9650	2.25	20.000	0.9351	0.9625	2.83

30.000	0.9614	0.9766	1.56	30.000	0.9558	0.9750	1.96
40.000	0.9707	0.9825	1.19	40.000	0.9665	0.9812	1.50
50.000	0.9764	0.9860	0.96	50.000	0.9730	0.9850	1.21
60.000	0.9803	0.9883	0.80	60.000	0.9774	0.9875	1.02
70.000	0.9830	0.9900	0.69	70.000	0.9805	0.9892	0.87
80.000	0.9851	0.9912	0.61	80.000	0.9829	0.9906	0.77
90.000	0.9867	0.9922	0.54	90.000	0.9848	0.9916	0.68
100.000	0.9881	0.9930	0.49	100.000	0.9863	0.9925	0.62
200.000	0.9940	0.9965	0.24	200.000	0.9931	0.9962	0.31
300.000	0.9960	0.9976	0.16	300.000	0.9954	0.9975	0.21
400.000	0.9969	0.9982	0.12	400.000	0.9965	0.9981	0.15
500.000	0.9975	0.9986	0.10	500.000	0.9972	0.9985	0.12
600.000	0.9979	0.9988	0.08	600.000	0.9976	0.9987	0.10
700.000	0.9982	0.9990	0.07	700.000	0.9980	0.9989	0.09
800.000	0.9984	0.9991	0.06	800.000	0.9982	0.9990	0.07
900.000	0.9986	0.9992	0.05	900.000	0.9984	0.9991	0.07

$p = 0.5$				$p = 0.75$			
u	f_{eC} (Equation 34)	f_{eC} (Equation 33)	$-(f_{eC} - f_{FS})/(f_{FS}(\%))$	u	f_{eC} (Equation 34)	f_{eC} (Equation 33)	$-(f_{eC} - f_{FS})/(f_{FS}(\%))$
0.001	0.0014	0.0014	3.82	0.001	0.0034	0.0034	0.68
0.002	0.0028	0.0029	3.82	0.002	0.0069	0.0069	0.68
0.003	0.0043	0.0044	3.82	0.003	0.0103	0.0103	0.68
0.004	0.0057	0.0059	3.81	0.004	0.0137	0.0138	0.68
0.005	0.0071	0.0074	3.81	0.005	0.0170	0.0172	0.68
0.006	0.0085	0.0089	3.81	0.006	0.0204	0.0205	0.68
0.007	0.0099	0.0103	3.81	0.007	0.0237	0.0239	0.68
0.008	0.0114	0.0118	3.81	0.008	0.0270	0.0272	0.68
0.009	0.0128	0.0133	3.81	0.009	0.0303	0.0305	0.68
0.010	0.0142	0.0147	3.81	0.010	0.0335	0.0338	0.68
0.020	0.0280	0.0291	3.81	0.020	0.0650	0.0654	0.68
0.030	0.0414	0.0431	3.81	0.030	0.0994	0.0950	0.68
0.040	0.0545	0.0567	3.81	0.040	0.1220	0.1229	0.68
0.050	0.0672	0.0699	3.81	0.050	0.1480	0.1490	0.68
0.060	0.0796	0.0828	3.81	0.060	0.1725	0.1737	0.68
0.070	0.0917	0.0953	3.80	0.070	0.1957	0.1970	0.68
0.080	0.1034	0.1075	3.80	0.080	0.2175	0.2190	0.68
0.090	0.1149	0.1194	3.80	0.090	0.2382	0.2399	0.68
0.100	0.1260	0.1310	3.80	0.100	0.2579	0.2597	0.68
0.200	0.2239	0.2327	3.78	0.200	0.4101	0.4129	0.68
0.300	0.3020	0.3138	3.76	0.300	0.5104	0.5139	0.67

TABLE IV Continued.

$p = 0.5$				$p = 0.75$			
u	f_{eC} (Equation 34)	f_{eC} (Equation 33)	$-(f_{eC} - f_{FS})/(f_{FS}(\%))$	u	f_{eC} (Equation 34)	f_{eC} (Equation 33)	$-(f_{eC} - f_{FS})/(f_{FS}(\%))$
0.400	0.3659	0.3801	3.74	0.400	0.5816	0.5856	0.67
0.500	0.4190	0.4352	3.72	0.500	0.6347	0.6390	0.67
0.600	0.4639	0.4818	3.70	0.600	0.6759	0.6804	0.67
0.700	0.5024	0.5216	3.67	0.700	0.7087	0.7135	0.66
0.800	0.5357	0.5561	3.65	0.800	0.7355	0.7404	0.66
0.900	0.5649	0.5861	3.62	0.900	0.7577	0.7628	0.66
1.000	0.5906	0.6126	3.60	1.000	0.7765	0.7817	0.66
2.000	0.7426	0.7681	3.32	2.000	0.8742	0.8797	0.62
3.000	0.8123	0.8375	3.01	3.000	0.9124	0.9177	0.57
4.000	0.8523	0.8761	2.72	4.000	0.9329	0.9377	0.52
5.000	0.8782	0.9003	2.45	5.000	0.9455	0.9500	0.47
6.000	0.8964	0.9167	2.21	6.000	0.9542	0.9583	0.42
7.000	0.9099	0.9286	2.01	7.000	0.9605	0.9642	0.39
8.000	0.9202	0.9375	1.83	8.000	0.9652	0.9687	0.35
9.000	0.9284	0.9444	1.68	9.000	0.9690	0.9722	0.32
10.000	0.9351	0.9500	1.56	10.000	0.9720	0.9750	0.30
20.000	0.9665	0.9750	0.87	20.000	0.9858	0.9875	0.17
30.000	0.9774	0.9833	0.87	30.000	0.9905	0.9916	0.11
40.000	0.9329	0.9875	0.45	40.000	0.9928	0.9937	0.08
50.000	0.9863	0.9900	0.37	50.000	0.9942	0.9950	0.07
60.000	0.9885	0.9916	0.31	60.000	0.9952	0.9958	0.06
70.000	0.9901	0.9928	0.26	70.000	0.9959	0.9964	0.05
80.000	0.9914	0.9937	0.23	80.000	0.9964	0.9968	0.04
90.000	0.9923	0.9944	0.20	90.000	0.9968	0.9972	0.04
100.000	0.9931	0.9950	0.18	100.000	0.9971	0.9975	0.03
200.000	0.9965	0.9975	0.09	200.000	0.9985	0.9987	0.01
300.000	0.9976	0.9983	0.06	300.000	0.9990	0.9991	0.01
400.000	0.9982	0.9987	0.04	400.000	0.9992	0.9993	0.00
500.000	0.9986	0.9990	0.03	500.000	0.9994	0.9995	0.00
600.000	0.9988	0.9991	0.03	600.000	0.9995	0.9995	0.00
700.000	0.9990	0.9992	0.02	700.000	0.9995	0.9996	0.00
800.000	0.9991	0.9993	0.02	800.000	0.9996	0.9996	0.00
900.000	0.9992	0.9994	0.02	900.000	0.9996	0.9997	0.00

TABLE V Compared values of the functions $f_{FS}(u)$ and $f_{eCL}(u)$ (Equations 33 and 43) respectively for low values of the electronic reflection coefficient p

$p = 0$			$p = 0.05$			$p = 0.1$		
u	f_{FS} (Equation 33)	f_{eCL} (Equation 43)	u	f_{FS} (Equation 33)	f_{eCL} (Equation 43)	u	f_{FS} (Equation 33)	f_{eCL} (Equation 43)
0.001	0.0004	0.0004	0.001	0.0005	0.0005	0.001	0.0006	0.0006
0.002	0.0009	0.0009	0.002	0.0011	0.0011	0.002	0.0012	0.0012
0.003	0.0014	0.0014	0.003	0.0016	0.0016	0.003	0.0018	0.0018
0.004	0.0019	0.0019	0.004	0.0022	0.0022	0.004	0.0024	0.0024
0.005	0.0024	0.0024	0.005	0.0027	0.0027	0.005	0.0030	0.0030
0.006	0.0029	0.0029	0.006	0.0033	0.0033	0.006	0.0036	0.0036
0.007	0.0034	0.0034	0.007	0.0038	0.0038	0.007	0.0042	0.0042
0.008	0.0039	0.0039	0.008	0.0044	0.0044	0.008	0.0048	0.0048
0.009	0.0044	0.0044	0.009	0.0049	0.0049	0.009	0.0054	0.0054
0.010	0.0049	0.0049	0.010	0.0055	0.0054	0.010	0.0060	0.0060
0.020	0.0099	0.0099	0.020	0.0109	0.0109	0.020	0.0121	0.0120
0.030	0.0148	0.0147	0.020	0.0163	0.0163	0.030	0.0180	0.0180
0.040	0.0197	0.0196	0.040	0.0217	0.0216	0.040	0.0239	0.0238
0.050	0.0245	0.0243	0.050	0.0270	0.0268	0.050	0.0298	0.0296
0.060	0.0294	0.0291	0.060	0.0323	0.0320	0.060	0.0356	0.0353
0.070	0.0431	0.0338	0.070	0.0376	0.0372	0.070	0.0413	0.0410
0.080	0.0389	0.0384	0.080	0.0428	0.0423	0.080	0.0470	0.0466
0.090	0.0436	0.0430	0.090	0.0479	0.0473	0.090	0.0527	0.0521
0.100	0.0483	0.0476	0.100	0.0531	0.0523	0.100	0.0583	0.0575
0.200	0.0936	0.0909	0.200	0.1022	0.0995	0.200	0.1115	0.1089
0.300	0.1360	0.1304	0.300	0.1476	0.1422	0.300	0.1602	0.1549
0.400	0.1758	0.1666	0.400	0.1898	0.1810	0.400	0.2049	0.1964
0.500	0.2130	0.2000	0.500	0.2290	0.2164	0.500	0.2460	0.2340
0.600	0.2480	0.2307	0.600	0.2654	0.2490	0.600	0.2839	0.2682
0.700	0.2808	0.2592	0.700	0.2993	0.2789	0.700	0.3189	0.2996
0.800	0.3116	0.2857	0.800	0.3310	0.3065	0.800	0.3513	0.3283
0.900	0.3406	0.3103	0.900	0.3606	0.3321	0.900	0.3814	0.3548
1.000	0.3678	0.3333	1.000	0.3882	0.3559	1.000	0.4093	0.3793
2.000	0.5676	0.5000	2.000	0.5864	0.5250	2.000	0.6055	0.5500
3.000	0.6832	0.6000	3.000	0.6983	0.6237	3.000	0.7135	0.6470
4.000	0.7545	0.6666	4.000	0.7666	0.6885	4.000	0.7787	0.7096
5.000	0.8013	0.7142	5.000	0.8112	0.7342	5.000	0.8210	0.7534
6.000	0.8337	0.7500	6.000	0.8420	0.7682	6.000	0.8503	0.7857
7.000	0.8572	0.7777	7.000	0.8644	0.7945	7.000	0.8715	0.8105
8.000	0.8750	0.8000	8.000	0.8812	0.8155	8.000	0.8875	0.8301
9.000	0.8889	0.8181	9.000	0.8944	0.8325	9.000	0.9000	0.8461
10.000	0.9000	0.8333	10.000	0.9050	0.8467	10.000	0.9100	0.8593

20.000	0.9500	0.9090	20.000	0.9525	0.9170	20.000	0.9525	0.9243
30.000	0.9666	0.9375	30.000	0.9683	0.9431	30.000	0.9700	0.9482
40.000	0.9750	0.9523	40.000	0.9762	0.9567	40.000	0.9775	0.9606
50.000	0.9800	0.9615	50.000	0.9810	0.9650	50.000	0.9820	0.9683
60.000	0.9833	0.9677	60.000	0.9841	0.9707	60.000	0.9850	0.9734
70.000	0.9857	0.9722	70.000	0.9864	0.9748	70.000	0.9871	0.9771
80.000	0.9875	0.9756	80.000	0.9881	0.9778	80.000	0.9887	0.9799
90.000	0.9888	0.9782	90.000	0.9894	0.9802	90.000	0.9900	0.9821

$p = 0.15$			$p = 0.2$			$p = 0.25$		
u	f_{FS} (Equation 33)	f_{eCL} (Equation 43)	u	f_{FS} (Equation 33)	f_{eCL} (Equation 43)	u	f_{FS} (Equation 33)	f_{eCL} (Equation 43)
0.001	0.0006	0.0006	0.001	0.0007	0.0007	0.001	0.0008	0.0008
0.002	0.0013	0.0013	0.002	0.0014	0.0014	0.002	0.0016	0.0016
0.003	0.0020	0.0020	0.003	0.0022	0.0022	0.003	0.0024	0.0024
0.004	0.0026	0.0026	0.004	0.0029	0.0029	0.004	0.0033	0.0033
0.005	0.0033	0.0033	0.005	0.0037	0.0037	0.005	0.0041	0.0041
0.006	0.0040	0.0040	0.006	0.0044	0.0044	0.006	0.0049	0.0049
0.007	0.0047	0.0047	0.007	0.0052	0.0052	0.007	0.0058	0.0057
0.008	0.0053	0.0053	0.008	0.0059	0.0059	0.008	0.0066	0.0066
0.009	0.0060	0.0060	0.009	0.0067	0.0067	0.009	0.0074	0.0074
0.010	0.0067	0.0067	0.010	0.0074	0.0074	0.010	0.0082	0.0082
0.020	0.0133	0.0133	0.020	0.0148	0.0147	0.020	0.0164	0.0163
0.030	0.0199	0.0198	0.030	0.0220	0.0220	0.030	0.0244	0.0243
0.040	0.0264	0.0263	0.040	0.0292	0.0291	0.040	0.0323	0.0322
0.050	0.0329	0.0327	0.050	0.0363	0.0361	0.050	0.0401	0.0400
0.060	0.0392	0.0390	0.060	0.0433	0.0430	0.060	0.0478	0.0476
0.070	0.0455	0.0452	0.070	0.0502	0.0498	0.070	0.0554	0.0551
0.080	0.0518	0.0513	0.080	0.0570	0.0566	0.080	0.0629	0.0625
0.090	0.0579	0.0573	0.090	0.0638	0.0632	0.090	0.0703	0.0697
0.100	0.0640	0.0633	0.100	0.0704	0.0697	0.100	0.0776	0.0769
0.200	0.1217	0.1191	0.200	0.1329	0.1304	0.200	0.1452	0.1428
0.300	0.1738	0.1687	0.300	0.1886	0.1836	0.300	0.2047	0.2000
0.400	0.2211	0.2129	0.400	0.2385	0.2307	0.400	0.2574	0.2500
0.500	0.2641	0.2527	0.500	0.2835	0.2727	0.500	0.3043	0.2941
0.600	0.3034	0.2887	0.600	0.3242	0.3103	0.600	0.3463	0.3333
0.700	0.3395	0.3213	0.700	0.3612	0.3442	0.700	0.3841	0.3684
0.800	0.3726	0.3511	0.800	0.3949	0.3750	0.800	0.4184	0.4000
0.900	0.4031	0.3784	0.900	0.4258	0.4029	0.900	0.4495	0.4285
1.000	0.4313	0.4035	1.000	0.4541	0.4285	1.000	0.4778	0.4545
2.000	0.6249	0.5750	2.000	0.6445	0.6000	2.000	0.6643	0.6250

TABLE V Continued

$p = 0.15$			$p = 0.2$			$p = 0.25$		
u	f_{FS} (Equation 33)	f_{eCL} (Equation 43)	u	f_{FS} (Equation 33)	f_{eCL} (Equation 43)	u	f_{FS} (Equation 33)	f_{eCL} (Equation 43)
3.000	0.7287	0.6699	3.000	0.7440	0.6923	3.000	0.7594	0.7142
4.000	0.7908	0.7301	4.000	0.8029	0.7500	4.000	0.8150	0.7692
5.000	0.8309	0.7718	5.000	0.8408	0.7894	5.000	0.8507	0.8064
6.000	0.8586	0.8023	6.000	0.8669	0.8181	6.000	0.8752	0.8333
7.000	0.8706	0.8256	7.000	0.8857	0.8400	7.000	0.8929	0.8536
8.000	0.8937	0.8440	8.000	0.9000	0.8571	8.000	0.9062	0.8695
9.000	0.9055	0.8589	9.000	0.9111	0.8709	9.000	0.9166	0.8823
10.000	0.9150	0.8712	10.000	0.9200	0.8823	10.000	0.9250	0.8928
20.000	0.9575	0.9311	20.000	0.9600	0.9375	20.000	0.9625	0.9433
30.000	0.9716	0.9530	30.000	0.9733	0.9574	30.000	0.9750	0.9615
40.000	0.9787	0.9643	40.000	0.9800	0.9677	40.000	0.9812	0.9708
50.000	0.9830	0.9712	50.000	0.9840	0.9740	50.000	0.9850	0.9765
60.000	0.9858	0.9759	60.000	0.9866	0.9782	60.000	0.9875	0.9803
70.000	0.9878	0.9793	70.000	0.9885	0.9813	70.000	0.9892	0.9831
80.000	0.9893	0.9818	80.000	0.9900	0.9836	80.000	0.9906	0.9852
90.000	0.9905	0.9838	90.000	0.9911	0.9854	90.000	0.9916	0.9868

TABLE VI Compared values of the F–S expression of the electrical conductivity (Equation 1), FUC, and of the e–C expression (Equation 6) using the new parameter μ^* (Equation 44), COT, for a series of values of the electronic reflection coefficient, p, and of the reduced thickness, k

$p = 0$				$p = 0.25$			
k	μ^*	FUC	COT	k	μ^*	FUC	COT
0.01	0.005	26.4830	27.7305	0.01	0.008	18.1950	18.5882
0.02	0.010	15.3320	16.1629	0.02	0.016	10.7600	1.0302
0.03	0.015	11.2540	11.9197	0.03	0.025	8.0187	8.2399
0.04	0.020	9.0885	9.6611	0.04	0.033	6.5548	6.7487
0.05	0.025	7.7276	8.2399	0.05	0.041	5.6317	5.8077
0.06	0.030	6.7859	7.2552	0.06	0.050	4.9909	5.1541
0.07	0.035	6.0917	6.5283	0.07	0.058	4.5174	4.6709
0.08	0.040	5.5566	5.9675	0.08	0.066	4.1516	4.2975
0.09	0.045	5.1302	5.5202	0.09	0.075	3.8597	3.9993
0.10	0.050	4.7816	5.1541	0.10	0.083	3.6206	3.7550
0.20	0.100	3.0958	3.3773	0.20	0.166	2.4593	2.5665
0.30	0.150	2.4658	2.7082	0.30	0.250	2.0230	2.1183
0.40	0.200	2.1284	2.3472	0.40	0.333	1.7889	1.8768
0.50	0.250	1.9161	2.1183	0.50	0.416	1.6415	1.7241
0.60	0.300	1.7695	1.9589	0.60	0.500	1.5398	1.6182
0.70	0.350	1.6621	1.8411	0.70	0.583	1.4652	1.5400
0.80	0.400	1.5798	1.7501	0.80	0.666	1.4081	1.4799
0.90	0.450	1.5148	1.6775	0.90	0.750	1.3630	1.4320
1.00	0.500	1.4622	1.6182	1.00	0.833	1.3265	1.3931
2.00	1.000	1.2208	1.3333	2.00	1.666	1.1585	1.2080
3.00	1.500	1.1414	1.2295	3.00	2.500	1.1026	1.1419
4.00	2.000	1.1031	1.1753	4.00	3.333	1.0754	1.1077
5.00	2.500	1.0810	1.1419	5.00	4.166	1.0595	1.0869
6.00	3.000	1.0666	1.1192	6.00	5.000	1.0491	1.0728
7.00	3.500	1.0566	1.1028	7.00	5.833	1.0418	1.0626
8.00	4.000	1.0491	1.0904	8.00	6.666	1.0364	1.0550
9.00	4.500	1.0434	1.0806	9.00	7.500	1.0322	1.0490
10.00	5.000	1.0389	1.0728	10.00	8.333	1.0289	1.0441

$p = 0.5$				$p = 0.75$			
k	μ^*	FUC	COT	k	μ^*	FUC	COT
0.01	0.015	11.8200	11.9197	0.01	0.035	6.5149	6.5283
0.02	0.030	7.1839	7.2552	0.02	0.070	4.1603	4.1706
0.03	0.045	5.4601	5.5202	0.03	0.105	3.2760	3.2851
0.04	0.060	4.5350	4.5888	0.04	0.140	2.7992	2.8076
0.05	0.075	3.9496	3.9993	0.05	0.175	2.4968	2.5047
0.06	0.090	3.5422	3.5889	0.06	0.210	2.2860	2.2937
0.07	0.105	3.2406	3.2851	0.07	0.245	2.1300	2.1374
0.08	0.120	3.0073	3.0501	0.08	0.280	2.0093	2.0166
0.09	0.135	2.8209	2.8622	0.09	0.315	1.9130	1.9201
0.10	0.150	2.6681	2.7082	0.10	0.350	1.8341	1.8411
0.20	0.300	1.9250	1.9589	0.20	0.700	1.4532	1.4595
0.30	0.450	1.6463	1.6775	0.30	1.050	1.3128	1.3188
0.40	0.600	1.4974	1.5268	0.40	1.400	1.2389	1.2447
0.50	0.750	1.4040	1.4320	0.50	1.750	1.1932	1.1987
0.60	0.900	1.3398	1.3667	0.60	2.100	1.1620	1.1674
0.70	1.050	1.2929	1.3188	0.70	2.450	1.1394	1.1446
0.80	1.200	1.2571	1.2821	0.80	2.800	1.1223	1.1274
0.90	1.350	1.2289	1.2531	0.90	3.150	1.1088	1.1138
1.00	1.500	1.2061	1.2295	1.00	3.500	1.0980	1.1028
2.00	3.000	1.1013	1.1192	2.00	7.000	1.0487	1.0524
3.00	4.500	1.0663	1.0806	3.00	10.500	1.0321	1.0351
4.00	6.000	1.0491	1.0609	4.00	17.500	1.0191	1.0212
5.00	7.500	1.0389	1.0490	4.00	14.000	1.0239	1.0264
6.00	9.000	1.0322	1.0409	6.00	21.000	1.0158	1.0177
7.00	10.500	1.0275	1.0351	7.00	24.500	1.0135	1.0152
8.00	12.000	1.0239	1.0308	8.00	28.000	1.0118	1.0133
9.00	13.500	1.0212	1.0274	9.00	31.500	1.0105	1.0118
10.00	15.000	1.0191	1.0247	10.00	35.000	1.0094	1.0106

the resultant mean free path, λ_r

$$\lambda_r = \lambda_0\left[1 + \frac{2(1-p)}{1+p}\frac{|\cos\theta|}{k}\right]^{-1} \qquad (42)$$

We then compare the numerical values of the function $f_{FS}(u)$ (Equation 33) and the function $f_{eCL}(u)$ defined by

$$f_{eCL}(u) = \left[1 + \frac{2(1-p)}{1+p}\frac{1}{u}\right]^{-1} \qquad (43)$$

Table V shows that the deviation is not marked, provided that $p < 0.3$.

As attempted, no marked deviation occurs in the expressions of the electrical resistivity in the frameworks of the F–S model and the e–C model (Table VI), in the case where the size parameter μ (Equation 7) is replaced by μ^* with

$$\mu^* = k\frac{1+p}{2(1-p)} \qquad (44)$$

Consequently, Equation 6 for $\mu = \mu^*$ holds for any value of p or d. Moreover it is clear that the substitution of μ^* for μ in the asymptotic Equation 9b (valid for $k \ll 1$) gives the usual Fuchs–Sondheimer expression (Equation4).

5. Conclusion

The above theoretical calculations and their numerical evaluations show that two formulations can be regarded as alternative forms of the Fuchs–Sondheimer size-effect function. The first function is the extended-Cottey size-effect function (Equations 6 and 7) whose validity is only restricted to the case $p > 0.31$ (without any restriction due to the film thickness). The second function has a general validity, whatever the roughness of the film surface and the film thickness: it is the Cottey size-effect function (Equation 5) where the parameter μ is replaced by μ^* (Equation 44); this expression is especially convenient at low thickness.

Alternative analytical forms are thus available for describing the Fuchs–Sondheimer size effects.

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